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Stochastic integral representation theorem for quantum semimartingales

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Abstract

The quantum stochastic integral of Itô type formulated by Hudson and Parthasarathy is extended to a wider class of adapted quantum stochastic processes on Boson Fock space. An Itô formula is established and a quantum stochastic integral representation theorem is proved for a class of unbounded semimartingales which includes polynomials and (Wick) exponentials of the basic martingales in quantum stochastic calculus.

Keywords: Boson Fock space; Quantum stochastic process; Quantum stochastic integral; Quantum Itô formula; Quantum martingale; Quantum semimartingale

1. Introduction

Since a quantum (non-commutative) stochastic calculus of Itô type first formulated by Hudson and Parthasarathy [20], the theory of quantum stochastic integration has been extensively developed in [3,6,25,29,36] and the references cited therein. The stochastic integral representations of quantum martingales which are quantum analogues of the classical Kunita and Watanabe theorem [21] have been studied by many authors (see [1,5,17,19,23,29,30,37,38], etc.). Such an integral representation theorem for non-Fock quantum Brownian motion with respect to the annihilation and creation processes was proved by Hudson and Lindsay [17]. In [37], Parthasarathy and Sinha established a stochastic integral representation of regular bounded quantum martingales in Fock space with respect to the basic martingales, namely the annihilation, creation and number processes. A new proof of the

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Parthasarathy and Sinha representation theorem has been discussed by Meyer [30] in which he gives the special form of the number operator coefficient. Also, the representation theorem has been extended to regular bounded semimartingales by Attal [1] and its applications have been discussed by many authors (see [2,18], etc). On the other hand, the annihilation, creation and number processes are not (bounded) regular in the sense of Parthasarathy and Sinha even though they have trivial integral representations.

Motivated by this fact, in this article by using the framework of Gaussian (white noise) analysis (see [13,14,22,31], etc; for the white noise approach to quantum probability, we refer to [8–10,16,32–35]), we first extend the Hudson and Parthasarathy quantum stochastic calculus and generalize the notions of regular martingale and semimartingale in the context of certain Gelfand triples and then we prove the integral representation theorems for regular unbounded quantum martingales and semimartingales. More precisely, let $\mathscr{H}$ be the Boson Fock space over $H = L^2_{\mathbb{C}}(\mathbb{R}_+)$ with norm $|||\cdot|||_0$ and let $\mathscr{G}_p$ be the weighted Fock space with norm $|||\cdot|||_p = |||e^{pN}\cdot|||_0$, $p\in\mathbb{R}$, where N is the number operator. Let $\mathscr{G}$ be the countable Hilbert space equipped with the Hilbertian norms $|||\cdot|||_p$ and $\mathscr{G}^*$ be the topological dual space of $\mathscr{G}$. Then we have a triplet $\mathscr{G}\subset\mathscr{H}\subset\mathscr{G}^*$ (see [7,12,39]; also [6,26,27]), one of the advantages of this triplet is that every integral kernel operator $\Xi_{l,m}(K_{l,m})$ with kernel $K_{l,m}\in\mathscr{L}(H^{\otimes m}, H^{\otimes l})$ is a bounded linear operator from $\mathscr{G}$ into itself and from $\mathscr{G}^*$ into itself. It follows that the basic martingales are quantum stochastic processes of bounded linear operators from $\mathscr{G}$ into itself and from $\mathscr{G}^*$ into itself (see Section 4). Based on the triplet, we first extend the Hudson and Parthasarathy quantum stochastic integral to a class of adapted processes of operators in $\mathscr{G}^*$ and the quantum Itô formula is derived in this setting. Secondly, we generalize the notions of regularity of bounded quantum martingales and semimartingales to certain quantum martingales and semimartingales of operators in $\mathscr{L}(\mathscr{G},\mathscr{G}^*)$, and then we prove integral representation theorems for these which are extensions of the integral representations of regular bounded martingales and semimartingales in Boson Fock space obtained by Parthasarathy and Sinha [37] and Attal [1], respectively. For our proofs we assume the continuum hypothesis (see Remark 7.7); the results may also be obtained under alternative assumptions (cf. [29]).

The paper is organized as follows. In Section 2 we briefly recall the space of regular (white noise) functions. In Section 3 we study generalized (classical) stochastic processes. In particular, we prove the stochastic integral representations of generalized (classical) martingales and semimartingales. In Section 4 we discuss the integral kernel operators on the space of regular functions and then the basic martingales are quantum stochastic processes of bounded linear operators acting on the space of regular functions. In Section 5 we extend the Hudson and Parthasarathy quantum stochastic integral to a wider class of adapted quantum stochastic processes of operators in $\mathscr{G}^*$ and discuss their properties. In Section 6 we derive the quantum Itô formula in this setting. In Section 7 we extend the notion of regularity of bounded quantum martingales to a certain class of quantum martingales of operators in $\mathscr{L}(\mathscr{G},\mathscr{G}^*)$ and establish their integral representation. In Section 8 we extend the notion of regularity of bounded quantum semimartingales to a certain class of quantum semimartingales of operators in $\mathscr{L}(\mathscr{G},\mathscr{G}^*)$ and establish their integral representation.

Notations. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally convex spaces, $\mathfrak{X}\otimes\mathfrak{Y}$ the Hilbert space tensor product when $\mathfrak{X}$ and $\mathfrak{Y}$ are Hilbert spaces, $L(\mathscr{D},\mathfrak{X})$ the space of all linear operators in $\mathfrak{X}$ with domain $\mathscr{D}$, and $\mathscr{L}(\mathfrak{X},\mathfrak{Y})$ the space of continuous linear operators in $\mathfrak{X}$ into $\mathfrak{Y}$; equipped with the topology of bounded convergence. For notational convenience, we write $\mathscr{L}(\mathfrak{X})\equiv\mathscr{L}(\mathfrak{X},\mathfrak{X})$.

2. Regular functions on Gaussian space

Let $H_{\mathbb{R}}=L^2_{\mathbb{R}}(\mathbb{R}_+)$ be the real Hilbert space of square integrable functions with respect to the Lebesgue measure dt on $\mathbb{R}_+=[0,\infty)$ and the norm denoted by $|\cdot|_0$. From $H_{\mathbb{R}}$ and a given positive self-adjoint operator A on $H_{\mathbb{R}}$ with $\rho\equiv\|A^{-1}\|_{OP}<1$ and $\delta\equiv\|A^{-1}\|_{\mathrm{HS}}<\infty$, a Gelfand triple $E\subset H_{\mathbb{R}}\subset E^*$ is constructed in the standard manner (see [14,22,31]). Then E is a nuclear space equipped with the Hilbertian norms $|\xi|_p=|A^p\xi|_0$, $p\in\mathbb{R}$. Let (E^*,μ) be the Gaussian space or white noise space with the standard Gaussian measure μ whose characteristic function is given by

$$\int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx)=\exp\left(-\frac{1}{2}|\xi|_0^2\right),\quad \xi\in E,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^*\times E$. Let $\mathscr{H}=L^2(E^*,\mu;\mathbb{C})$ be the complex Hilbert space of square integrable functions with respect to μ on E^* and the norm denoted by $|||\cdot|||_0$. Then by the Wiener–Itô decomposition theorem, we have the following unitary isomorphism between $\mathscr{H}$ and the Boson Fock space $\Gamma(H)$:

$$\mathscr{H}\ni\phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,f_n\rangle \quad\leftrightarrow\quad (f_n)\sim\phi\in\Gamma(H),\quad f_n\in H^{\hat{\otimes}n},$$

where $:x^{\otimes n}:$ denotes the Wick ordering of $x^{\otimes n}$ (see [22,31]) and $H^{\hat{\otimes}n}$ is the n-fold symmetric tensor product of H. Moreover, the $\mathscr{H}$-norm $|||\phi|||_0$ of $\phi\in\mathscr{H}$ is given by

$$|||\phi|||_0^2=\sum_{n=0}^{\infty}n!|f_n|_0^2,\quad \phi\sim(f_n),$$

where $|\cdot|_0$ denotes also the $H^{\otimes n}$-norm for any n.

Let N be the number operator and let $\mathscr{G}_p$ be the $\mathscr{H}$-domain of e^{pN} for each $p\geqslant 0$. Then $\mathscr{G}_p$ is a Hilbert space with norm $|||\cdot|||_p:=|||e^{pN}\cdot|||_0$. More precisely, for any $p \geqslant 0$

$$|||\phi|||_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2, \quad \phi \sim (f_n) \in \mathscr{G}_p.$$

Let $\mathscr{G}$ be the projective limit of $\{\mathscr{G}_p; p \geqslant 0\}$ and $\mathscr{G}^*$ be the topological dual space of $\mathscr{G}$. Then $\mathscr{G}^*$ is isomorphic to the inductive limit of $\{\mathscr{G}_{-p}; p \geqslant 0\}$, where $\mathscr{G}_{-p} = \mathscr{G}_p^*$ is the Hilbert space with norm $|||\cdot|||_{-p}$. In fact, for each $p \geqslant 0$, $\mathscr{G}_{-p}$ is the completion of $\mathscr{H}$ with respect to the norm $|||\cdot|||_{-p}$. Then we have a triplet

$$\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*.$$

An element in $\mathscr{G}$ (and in $\mathscr{G}^*$) is called a *regular test* (*and generalized, respectively*) *function* (see [7,12,39]; also [6,26,27]). The canonical bilinear form on $\mathscr{G}^* \times \mathscr{G}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Note that for any $p \in \mathbb{R}$, $e^{pN}\mathscr{H} = \mathscr{G}_{-p}$ and $e^{-pN}\mathscr{G}_{-p} = \mathscr{H}$. Moreover, $e^{pN}\mathscr{G}_q = \mathscr{G}_{q-p}$ for any $p, q \in \mathbb{R}$.

Let $\xi_B = \xi\chi_B$ for $B \subset \mathbb{R}_+$ and the indicator function χ_B. For notational convenience we write $\xi_{t]} = \xi_{[0,t]}$ and $\xi_{[t} = \xi_{[t,\infty)}$ for any $t \in \mathbb{R}_+$. Then for any $0 < s < t < \infty$, we have the decomposition

$$H = H_{s]} \oplus H_{[s,t]} \oplus H_{[t},$$

where $H_B = \{\xi_B; \xi \in H\}$, with abbreviations $H_{s]}$ and $H_{[t}$ when $B = [0, s]$, respectively, $[t, \infty)$. Therefore, we have the identification

$$\mathscr{H} = \mathscr{H}_{s]} \otimes \mathscr{H}_{[s,t]} \otimes \mathscr{H}_{[t}$$

via the following decomposition:

$$\phi_\xi = \phi_{\xi_{s]}} \otimes \phi_{\xi_{[s,t]}} \otimes \phi_{\xi_{[t}}, \quad \xi \in H,$$

where $\mathscr{H}_{s]} \cong \Gamma(H_{s]})$, $\mathscr{H}_{[s,t]} \cong \Gamma(H_{[s,t]})$, $\mathscr{H}_{[t} \cong \Gamma(H_{[t]})$ and $\phi_\xi \sim (\xi^{\otimes n}/n!)$ is the exponential vector of $\xi \in H$. Moreover, for any $p \in \mathbb{R}$ and $0 < s < t < \infty$, we have

$$\mathscr{G}_p = \mathscr{G}_{p;s]} \otimes \mathscr{G}_{p;[s,t]} \otimes \mathscr{G}_{p;[t},$$

where $\mathscr{G}_{p;s]} = \mathscr{G}_p \cap \mathscr{H}_{s]}$, $\mathscr{G}_{p;[s,t]} = \mathscr{G}_p \cap \mathscr{H}_{[s,t]}$ and $\mathscr{G}_{p;[t} = \mathscr{G}_p \cap \mathscr{H}_{[t}$ (closures when $p < 0$).

3. Generalized stochastic processes

The Brownian motion B_t on E^* is defined by

$$B_t(x) = \lim_{n \to \infty} \langle x, \xi_n \rangle \quad \text{in } \mathscr{H},$$

where for each $t \geqslant 0$, $\{\xi_n\}_{n=1}^{\infty} \subset E$ is a sequence converging to the indicator function $\chi_{t]} \equiv \chi_{[0,t]}$ in $H_{\mathbb{R}}$. For each $t \geqslant 0$, let $\mathscr{F}_t$ be the σ-field generated by $\{B_s; 0 \leqslant s \leqslant t\}$.

A family $\Phi = \{\Phi_t\}_{t \geqslant 0}$ is called a *generalized stochastic process* [7] if there exists a $p \geqslant 0$ (independent of $t \geqslant 0$) such that $\Phi_t \in \mathscr{G}_{-p}$ for all $t \geqslant 0$ and the map $t \mapsto \Phi_t \in \mathscr{G}_{-p}$ is Borel measurable on $\mathbb{R}_+$. A generalized stochastic process $\{\Phi_t \sim (F_{t;n})\}_{t \geqslant 0}$ is said to be *adapted* (w.r.t. $\mathscr{F}_t$) if for all $t \geqslant 0$ and $n \geqslant 0$, $\operatorname{supp} F_{t;n} \subset [0,t]^n$.

Proposition 3.1 (Benth and Potthoff [7]). *Let $p \in \mathbb{R}$ and let Φ be an (generalized) adapted process in $\mathscr{G}_p$ such that $\int_0^t |||\Phi_s|||_p^2 \, ds < \infty$ for all $t \geqslant 0$. Then*

$$\int_0^t \Phi_s \, dB_s = e^{-p(N-1)} \int_0^t (e^{pN} \Phi_s) \, dB_s, \quad t \geqslant 0. \tag{3.1}$$

If $p < 0$, then the integral on the left-hand side of (3.1) is the generalized Itô integral [7]. By (3.1) and isometry of Itô integral,

$$\left|\left|\left| \int_0^t \Phi_s \, dB_s \right|\right|\right|_p^2 = e^{2p} \int_0^t |||\Phi_s|||_p^2 \, ds. \tag{3.2}$$

For each $t \in \mathbb{R}_+$, the *conditional expectation* $\mathbf{E}_t$ (w.r.t. $\mathscr{F}_t$) is defined by the second quantization operator $\Gamma(\chi_{t]})$ of $\chi_{t]}$, i.e., for each $t \in \mathbb{R}_+$

$$\mathbf{E}_t \Phi \sim (\chi_{t]}^{\otimes n} f_n), \quad \Phi \sim (f_n) \in \mathscr{G}^*.$$

Then for any $p \in \mathbb{R}$ and $\Phi \sim (f_n) \in \mathscr{G}_p$, we have

$$|||\mathbf{E}_t \Phi|||_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |\chi_{t]}^{\otimes n} f_n|_0^2 \leqslant |||\Phi|||_p^2.$$

Hence for any $p \in \mathbb{R}$ and $t \in \mathbb{R}_+$, $\mathbf{E}_t \in \mathscr{L}(\mathscr{G}_p)$ and $\mathbf{E}_t$ is an orthogonal projection. Moreover, $\mathbf{E}_t \in \mathscr{L}(\mathscr{G})$ and $\mathbf{E}_t \in \mathscr{L}(\mathscr{G}^*)$. An adapted process Φ in $\mathscr{G}^*$ is called a *martingale* if $\mathbf{E}_s \Phi_t = \Phi_s$ for any $0 \leqslant s \leqslant t$.

Theorem 3.2. *Let Φ be a martingale in $\mathscr{G}^*$. Then there is a unique adapted process Ψ in $\mathscr{G}^*$ such that*

$$\Phi_t = \Phi_0 + \int_0^t \Psi_s \, dB_s, \quad t \geqslant 0. \tag{3.3}$$

Proof. Without lose of generality, we assume that $\Phi_0 = 0$. Choose p such that Φ is a martingale in $\mathscr{G}_p$. Put $\Phi'_t = e^{pN} \Phi_t$, $t \geqslant 0$. Then Φ' is a martingale in $\mathscr{H}$ with $\Phi'_0 = 0$. By the Kunita–Watanabe theorem (see [21]), there exists an adapted process Ψ' in $\mathscr{H}$

such that

$$\Phi'_t = \int_0^t \Psi'_s \, dB_s, \quad t \geqslant 0.$$

Therefore, by Proposition 3.1, we have

$$\Phi_t = e^{-pN} \Phi'_t = \int_0^t e^{-p(N+1)} \Psi'_s \, dB_s.$$

By setting $\Psi_t = e^{-p(N+1)} \Psi'_t \in \mathscr{G}_p$, the representation in (3.3) is obvious. Suppose that there is also a process Ψ'' in $\mathscr{G}_{p-r}$, where $r \geqslant 0$, such that

$$\Phi_t = \Phi_0 + \int_0^t \Psi''_s \, dB_s, \quad t \geqslant 0.$$

It follows from (3.3) that

$$\int_0^t e^{(p-r)(N+1)} \Psi_s \, dB_s = e^{(p-r)N} (\Phi_t - \Phi_0) = \int_0^t e^{(p-r)(N+1)} \Psi''_s \, dB_s.$$

Therefore, by the uniqueness of the Kunita–Watanabe theorem, $\Psi_t = \Psi''_t$. □

An adapted process Φ in $\mathscr{G}_p$ is called a *regular semimartingale* if Φ admits a unique decomposition as

$$\Phi_t = M_t + \int_0^t \Psi_s \, ds \tag{3.4}$$

for a martingale M in $\mathscr{G}_p$ and an adapted process Ψ in $\mathscr{G}_p$ with $\int_0^t |||\Psi_s|||_p \, ds < \infty$.

Theorem 3.3. *Let Φ be an adapted process in $\mathscr{G}_p$. Then Φ is a regular semimartingale if and only if there exists a locally integrable (non-negative) function h on $\mathbb{R}_+$ such that for any $s \leqslant t$*

$$|||\mathbf{E}_s(\Phi_t) - \Phi_s|||_p \leqslant \int_s^t h(u) \, du.$$

In this case, Φ has the integral representation in (3.4) *with a martingale M and an adapted process Ψ satisfying $|||\Psi_u|||_p \leqslant h(u)$.*

Proof. If Φ is a regular semimartingale with decomposition (3.4), then the estimate is obvious by taking $h(u) = |||\Psi_u|||_p$ for any $u \geqslant 0$. The converse is a simple application of Enchev's characterization of Hilbertian quasimartingales [11] (see [2,4,28]). □

4. Integral kernel operators on regular functions

Let l, m be non-negative integers.

Lemma 4.1. *Let $K_{l,m} \in \mathscr{L}(H^{\otimes m}, H^{\otimes l})$. Then for any $p \in \mathbb{R}$ and $q > 0$, there exist constants $C, D_q \geqslant 0$ such that for any $\phi \sim (f_n) \in \mathscr{G}$*

$$\sum_{n=0}^{\infty} (l+n)! e^{2p(l+n)} |g_{l+n}|_0^2 \leqslant C^2 (e^{2(pl-(p+q)m)+q}) l^l m^m D_q^{(l+m)} |||\phi|||_{p+q}^2,$$

where $g_{l+n} = (((n+m)!/n!)(K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}})$ and $(f)_{\mathrm{sym}}$ is the symmetrization of $f \in H^{\otimes m}$.

Proof. Since $K_{l,m} \in \mathscr{L}(H^{\otimes m}, H^{\otimes l})$, there exists $C \geqslant 0$ such that for any $f \in H^{\otimes m}$, $|K_{l,m} f|_0 \leqslant C|f|_0$. It follows from the fact $|(f)_{\mathrm{sym}}|_0 \leqslant |f|_0$ that

$$|(K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}}|_0 \leqslant |K_{l,m} \otimes I^{\otimes n} f_{n+m}|_0 \leqslant C |f_{n+m}|_0.$$

Hence for any $p \in \mathbb{R}$, $q > 0$ and $\phi \sim (f_n) \in \mathscr{G}$ we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} (l+n)! e^{2p(l+n)} |g_{l+n}|_0^2 &\leqslant C^2 \sum_{n=0}^{\infty} (n+m)! \frac{(l+n)!}{n!} \frac{(n+m)!}{n!} e^{2p(l+n)} |f_{n+m}|_0^2 \\ &\leqslant C^2 e^{2(pl-(p+q)m)} \max_{n \geqslant 0} \left\{ \frac{(l+n)!}{n!} \frac{(n+m)!}{n!} e^{-2qn} \right\} |||\phi|||_{p+q}^2.\end{aligned}$$

Note that for any $s > 0$ and integer $m \geqslant 0$,

$$\max_{x \geqslant 0} (x+m) \cdots (x+1) e^{-sx} \leqslant e^{s/2} m^m D_s^m, \quad D_s = \frac{e^{s/2}}{se}. \tag{4.1}$$

Therefore, we have that for any $p \in \mathbb{R}$ and $q > 0$

$$\sum_{n=0}^{\infty} (l+n)! e^{2p(l+n)} |g_{l+n}|_0^2 \leqslant C^2 (e^{2(pl-(p+q)m)+q}) l^l m^m D_q^{l+m} |||\phi|||_{p+q}^2.$$

Thus the proof is complete. □

For each $K_{l,m} \in \mathscr{L}(H^{\otimes m}, H^{\otimes l})$, by Lemma 4.1 we define a linear operator $\Xi_{l,m}(K_{l,m})$ on $\mathscr{G}$ by

$$\Xi_{l,m}(K_{l,m})\phi \sim \left(\frac{(n+m)!}{n!} (K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}} \right), \quad \phi \sim (f_n) \in \mathscr{G}. \tag{4.2}$$

Now the following theorem is obvious.

Theorem 4.2. *Let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$ with operator norm C. Then $\Xi_{l,m}(K_{l,m})$ defined as in* (4.2) *is a continuous linear operator on $\mathcal{G}$. Moreover, for any $p \in \mathbb{R}$ and $q > 0$ we have*

$$|||\Xi_{l,m}(K_{l,m})\phi|||_p \leqslant C(e^{pl-(p+q)m+q/2})l^{l/2}m^{m/2}D_q^{(l+m)/2}|||\phi|||_{p+q}, \quad \phi \in \mathcal{G}. \tag{4.3}$$

The operator $\Xi_{l,m}(K_{l,m})$ is called the *integral kernel operator* with kernel $K_{l,m}$ (see [15]).

By duality and (4.3), the following proposition is obvious.

Proposition 4.3. *Let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$ with operator norm C. Then for any $p \geqslant 0$ and $q > 0$, we have*

$$|||\Xi_{l,m}(K_{l,m})\phi|||_{-(p+q)} \leqslant C(e^{pm-(p+q)l+q/2})l^{l/2}m^{m/2}D_q^{(l+m)/2}|||\phi|||_{-p}, \quad \phi \in \mathcal{G}.$$

Theorem 4.4. *Let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$. Then $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G})$ has a unique extension to a continuous linear operator from $\mathcal{G}^*$ into itself, which we also denote $\Xi_{l,m}(K_{l,m})$.*

Proof. The proof is obvious from Proposition 4.3. $\square$

Example 4.5. Let $\eta \in H$ and let $K_\eta \in \mathcal{L}(H, \mathbb{C})$ be defined by $K_\eta(f) = \langle \eta, f \rangle$ for any $f \in H$. For simple notation, we identify $\eta = K_\eta = K_\eta^*$, where K_η^* is the adjoint operator of K_η, i.e., $K_\eta^*(a) = a\eta$ for all $a \in \mathbb{C}$. Then the *annihilation operator* a_η and the *creation operator* a_η^* associated with η are defined by $a_\eta = \Xi_{0,1}(\eta)$ and $a_\eta^* = \Xi_{1,0}(\eta)$, respectively, and

$$a_\eta \in \mathcal{L}(\mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*), \quad a_\eta^* \in \mathcal{L}(\mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*).$$

For each $t \geqslant 0$, we put

$$A_t = \Xi_{0,1}(\chi_{t]}), \quad A_t^* = \Xi_{1,0}(\chi_{t]}), \quad \Lambda_t = \Xi_{1,1}(\chi_{t]}).$$

For the definition of Λ_t, the indicator function $\chi_{t]}$ is considered as a multiplication operator on H, i.e., $\chi_{t]}(\xi) = \xi_{t]}$ for any $\xi \in H$. Then for each $t \geqslant 0$, A_t, A_t^*, $\Lambda_t \in \mathcal{L}(\mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*)$. The processes $\{A_t\}_{t\geqslant 0}$, $\{A_t^*\}_{t \geqslant 0}$ and $\{\Lambda_t\}_{t\geqslant 0}$ are called the *annihilation*, *creation* and *number* (*or gauge*) processes, respectively.

For two integral kernel operators $\Xi_{l_1,m_1}(K)$ and $\Xi_{l_2,m_2}(T)$ the *Wick product or normal-ordered product* is defined by

$$\Xi_{l_1,m_1}(K) \diamond \Xi_{l_2,m_2}(T) = \Xi_{l_1+l_2,m_1+m_2}(K \otimes T).$$

Then for any $s, t \geqslant 0$ we have

$$A_s \diamond A_t = A_s A_t, \quad A_s^* \diamond A_t = A_s^* A_t, \quad A_s \diamond A_t^* = A_t^* A_s, \quad A_s^* \diamond A_t^* = A_s^* A_t^*$$

and

$$\Lambda_s \diamond \Lambda_t = \Lambda_s \Lambda_t - \Lambda_{s \wedge t}, \quad s \wedge t = \min\{s, t\}.$$

For each $a, b \in \mathbb{C}$, let $Q_{a,b}(t) = aA_t + bA_t^*, t \geqslant 0$. The quantum stochastic process Q, given by $Q_t = Q_{1,1}(t) = A_t + A_t^*$, is called *quantum Brownian motion* or *position process*. Then for any positive integer n, we have

$$Q_{a,b}^{\diamond n}(t) = \overbrace{Q_{a,b}(t) \diamond \cdots \diamond Q_{a,b}(t)}^{n\text{-times}} = \sum_{l+m=n} \frac{n!}{l!m!} a^m b^l (A_t^*)^l A_t^m, \quad t \geqslant 0 \tag{4.4}$$

and, by Theorem 4.2, for any $p \in \mathbb{R}$ and $q > 0$

$$|||(A_t^*)^l A_t^m \phi|||_p \leqslant (e^{pl-(p+q)m+q/2}) l^{l/2} m^{m/2} (tD_q)^{(l+m)/2} |||\phi|||_{p+q}, \quad \phi \in \mathscr{G}.$$

Therefore, by Stirling's formula, we can easily see that for any $p \in \mathbb{R}$ and $q > 0$ there exists a constant $C_{a,b;p,q}(t) \geqslant 0$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{l+m=n} \frac{n!}{l!m!} |a|^m |b|^l |||(A_t^*)^l A_t^m \phi|||_p \right) \leqslant C_{a,b;p,q}(t) |||\phi|||_{p+q}, \quad \phi \in \mathscr{G}.$$

It follows that for any $a, b \in \mathbb{C}$ and $t \geqslant 0$ the series

$$\operatorname{wexp} Q_{a,b}(t) := \sum_{n=0}^{\infty} \frac{1}{n!} Q_{a,b}^{\diamond n}(t)$$

converges in $\mathscr{L}(\mathscr{G})$. Moreover, for any $p \in \mathbb{R}$ and $q > 0$, $\operatorname{wexp} Q_{a,b}(t) \in \mathscr{L}(\mathscr{G}_{p+q}, \mathscr{G}_p)$ and by the same argument of that used in the proof of Theorem 4.4, we have $\operatorname{wexp} Q_{a,b}(t) \in \mathscr{L}(\mathscr{G}^*)$. The operator $\operatorname{wexp} Q_{a,b}(t)$ is called the *Wick exponential* of $Q_{a,b}(t)$ (see [9,10,35]). By similar arguments, we can prove that for any $p \in \mathbb{R}$ and $q > \lambda_0 > 0$, where λ_0 satisfies the equation $e^{-\lambda_0} = \lambda_0^2$, $\operatorname{wexp} \Lambda_t \in \mathscr{L}(\mathscr{G}_{p+q}, \mathscr{G}_p)$ for any $t \geqslant 0$ and $\operatorname{wexp} \Lambda_t \in \mathscr{L}(\mathscr{G}) \cap \mathscr{L}(\mathscr{G}^*)$. On the other hand, by Proposition 4.1 in [8], for any $t \geqslant 0$ and positive integer n

$$\Lambda_t^{\diamond (n+1)} = \Lambda_t^{\diamond n} (\Lambda_t - nI).$$

Therefore,

$$\Lambda_t^{\diamond (n+1)} = \Lambda_t (\Lambda_t - I)(\Lambda_t - 2I) \ldots (\Lambda_t - nI).$$

Hence

$$e^{\Lambda_t} = \mathrm{wexp}\,((e-1)\Lambda_t),$$

where we used the fact that

$$e^x = \sum_{n=0}^{\infty} \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!}(e-1)^n.$$

5. Quantum stochastic processes and quantum stochastic integral

Let $L^2_{\mathrm{lb}}(\mathbb{R}_+) \subset H$ be the subspace of locally bounded functions. We denote by $\mathscr{E}_{\mathrm{lb}}$ and $\mathscr{E}_{\mathrm{lb};t]}$ the dense subspaces of $\mathscr{H}$ and $\mathscr{H}_{t]}$ spanned by the exponential vectors ϕ_ξ and $\phi_{\xi_{t]}}$, respectively, with $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$. Also, we denote by $\mathscr{E}$ the dense subspace of $\mathscr{H}$ spanned by the exponential vectors ϕ_ξ with $\xi \in H$. Note that $\mathscr{E} \subset \mathscr{G}$.

In this section, by using similar arguments to those of [20] we extend the quantum stochastic integral with respect to the basic martingales introduced by Hudson and Parthasarathy to a wider class of adapted quantum stochastic processes. A family of operators $\Xi = \{\Xi_t\}_{t \geqslant 0} \subset L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}^*)$ is called a *quantum stochastic process* if there exists $p \in \mathbb{R}$ (independent of $t \geqslant 0$) such that $\Xi_t \in L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ for each $t \geqslant 0$ and for each $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ the map $t \mapsto \Xi_t \phi_\xi$ is strongly measurable. We may then think of Ξ_t as a densely defined operator on the Hilbert space $\mathscr{G}_p$; and call Ξ *adapted* if $\Xi_t = \Xi_{t]} \otimes_{\mathrm{alg}} I_{[t}$ for some $\Xi_{t]} \in L(\mathscr{E}_{\mathrm{lb};t]}, \mathscr{G}_{p;t]})$, where $I_{[t}$ is the identity operator on $\mathscr{G}_{p;[t}$. We speak of adapted processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$; such a process is called a *quantum martingale* if for any $0 \leqslant s \leqslant t$ and $\xi, \eta \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$

$$\langle\!\langle \Xi_t \phi_{\xi_{s]}}, \phi_{\eta_{s]}} \rangle\!\rangle = \langle\!\langle \Xi_s \phi_{\xi_{s]}}, \phi_{\eta_{s]}} \rangle\!\rangle.$$

For any non-negative integers l, m, the processes $\{(A_t^*)^l A_t^m\}_{t \geqslant 0}$ and $\{\Lambda_t^{\diamond l}\}_{t \geqslant 0}$ are quantum martingales. In particular, the annihilation process $\{A_t\}_{t \geqslant 0}$, creation process $\{A_t^*\}_{t \geqslant 0}$ and number process $\{\Lambda_t\}_{t \geqslant 0}$ are quantum martingales.

Definition 5.1. Let Ξ be an adapted process in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$. Then Ξ is said to be

(i) *simple* if there exists an increasing sequence $\{t_n;\ n = 0, 1, \ldots$ with $t_0 = 0$ and $t_n \to \infty$ as $n \to \infty\}$ such that $\Xi_t = \sum_{n=0}^{\infty} \Xi_n \chi_{[t_n, t_{n+1})}(t)$, where $\Xi_n = \Xi_{t_n}$,

(ii) *continuous* if for each $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ the map $t \mapsto \Xi_t \phi_\xi$ is strongly continuous $\mathbb{R}_+ \to \mathscr{G}_p$,

(iii) *locally square integrable* if for each $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$

$$\int_0^t |||\Xi_s \phi_\xi|||_p^2 \, ds < \infty \quad \text{for all } t \geqslant 0. \tag{5.1}$$

Note that condition (5.1) is equivalent to the condition that for each $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ the map $t \mapsto \Xi_{t]}\phi_\xi$ is locally square integrable.

By Theorem 4.2 the annihilation and creation processes are continuous (and therefore locally square integrable) processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ for every p. On the other hand, by similar arguments to those used in the proof of Lemma 4.1, for any $p \in \mathbb{R}$ and $t > s \geqslant 0$

$$|||(\Lambda_t - \Lambda_s)\phi_\xi|||_p^2 \leqslant e^{2p+1}\left(\int_s^t |\xi(u)|^2 du\right)|||\phi_\xi|||_{p+1/2}^2, \quad \xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$$

and hence the number process is continuous too.

For each $p \in \mathbb{R}$ and any $t \geqslant 0$, we define $T_{p;t} : \mathscr{G}^* \to \mathscr{G}^*$ by $T_{p;t}\phi = \left(e^{pN}_{t]} \otimes I_{q;[t}\right)\phi$ for ϕ in $\mathscr{G} = \mathscr{G}_{q;t]} \otimes \mathscr{G}_{q;[t}$. Obviously, $T_{-p;t}T_{p;t} = T_{p;t}T_{-p;t} = I$.

Proposition 5.2. *Let Ξ be a locally square integrable process in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$. Then there exists a sequence $\{\Xi^{(n)}\}_{n=1}^\infty$ of simple processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ such that for each $t \geqslant 0$ and $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$,*

$$\lim_{n\to\infty} \int_0^t |||(\Xi_s - \Xi_s^{(n)})\phi_\xi|||_p^2 \, ds = 0.$$

Proof. Suppose that Ξ satisfies (5.1). Then $\{T_{p;t}\Xi_t\}_{t\geqslant 0}$ satisfies (5.1) with the $\mathscr{H}$-norm $|||\cdot|||_0$. Therefore, by Proposition 3.2 in [20], there exists a sequence of simple processes $\{F^{(n)}\}_{n=1}^\infty$ in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{H})$ such that for each $t \geqslant 0$ and $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$,

$$\lim_{n\to\infty} \int_0^t |||(T_{p;s}\Xi_s - F_s^{(n)})\phi_\xi|||_0^2 \, ds = 0.$$

Hence the proof is obvious by setting $\Xi_t^{(n)} = T_{-p;t}F_t^{(n)}$ for any $t \geqslant 0$ and $n = 1, 2, \dots$. $\square$

Definition 5.3. Let E, F, G, H be simple processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ with

$$E_t = \sum_{n=0}^\infty E_n \chi_{[t_n, t_{n+1})}(t), \quad F_t = \sum_{n=0}^\infty F_n \chi_{[t_n, t_{n+1})}(t),$$
$$G_t = \sum_{n=0}^\infty G_n \chi_{[t_n, t_{n+1})}(t), \quad H_t = \sum_{n=0}^\infty H_n \chi_{[t_n, t_{n+1})}(t),$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n \to \infty$. Family of operators $\Xi = \{\Xi_t\}_{t\geqslant 0}$ in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ defined by $\Xi_0 = 0$ and

$$\Xi_t = \Xi_{t_n} + E_n(\Lambda_t - \Lambda_{t_n}) + F_n(A_t - A_{t_n}) + G_n(A_t^* - A_{t_n}^*) + H_n(t - t_n) \tag{5.2}$$

for $t_n \leqslant t < t_{n+1}$, is called the *quantum stochastic integral* of (E, F, G, H) against Λ, A, A^* and the time process, and we write

$$\Xi_t = \int_0^t (E\, d\Lambda + F\, dA + G\, dA^* + H\, ds).$$

Let E, F, G, H be simple processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ and let Ξ be their quantum stochastic integral. Then by direct computation for any $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$, $\eta \in H$ and $t \geqslant 0$

$$\langle\!\langle \Xi_t \phi_\xi, \phi_\eta \rangle\!\rangle = \int_0^t \langle\!\langle [\xi(s)\eta(s)E_s + \xi(s)F_s + \eta(s)G_s + H_s]\phi_\xi, \phi_\eta \rangle\!\rangle\, ds. \tag{5.3}$$

Moreover, for any $0 \leqslant s < t$, $\phi \in \mathscr{G}_{-p;s]}$, $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ and $\eta \in H$

$$\begin{aligned} &\langle\!\langle (\Xi_t - \Xi_s)\phi_\xi, \phi \otimes \phi_{\eta_{[s}} \rangle\!\rangle \\ &\quad = \int_s^t \langle\!\langle [\xi(u)\eta(u)E_u + \xi(u)F_u + \eta(u)G_u + H_u]\phi_\xi, \phi \otimes \phi_{\eta_{[s}} \rangle\!\rangle\, du, \end{aligned} \tag{5.4}$$

where if $p \geqslant 0$, then $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is considered as the canonical bilinear form on $\mathscr{G} \times \mathscr{G}^*$.

By similar arguments to those used in [20] or [36], we have the following theorem.

Theorem 5.4. *Let $E, E', F, F', G, G', H, H'$ be simple processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ and let*

$$\begin{aligned} \Xi_t &= \int_0^t (E\, d\Lambda + F\, dA + G\, dA^* + H\, ds), \\ \Xi'_t &= \int_0^t (E'\, d\Lambda + F'\, dA + G'\, dA^* + H'\, ds). \end{aligned}$$

Then Ξ and Ξ' are processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ and for any $\xi, \eta \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ we have

$$\begin{aligned} \langle\!\langle \Xi_t \phi_\xi, \Xi'_t \phi_\eta \rangle\!\rangle_p = \int_0^t \{ &\langle\!\langle \Xi_s \phi_\xi, [e^{2p}\xi(s)\eta(s)E'_s + \eta(s)F'_s + e^{2p}\xi(s)G'_s + H'_s]\phi_\eta \rangle\!\rangle_p \\ &+ \langle\!\langle [e^{2p}\xi(s)\eta(s)E_s + \xi(s)F_s + e^{2p}\eta(s)G_s + H_s]\phi_\xi, \Xi'_s\phi_\eta \rangle\!\rangle_p \\ &+ e^{2p}\xi(s)\eta(s)\langle\!\langle E_s\phi_\xi, E'_s\phi_\eta \rangle\!\rangle_p + e^{2p}\xi(s)\langle\!\langle E_s\phi_\xi, G'_s\phi_\eta \rangle\!\rangle_p \\ &+ e^{2p}\eta(s)\langle\!\langle G_s\phi_\xi, E'_s\phi_\eta \rangle\!\rangle_p + e^{2p}\langle\!\langle G_s\phi_\xi, G'_s\phi_\eta \rangle\!\rangle_p \}\, ds, \end{aligned} \tag{5.5}$$

where $\langle\!\langle \Xi_t\phi_\xi, \Xi'_t\phi_\eta \rangle\!\rangle_p = \langle\!\langle e^{pN}\Xi_t\phi_\xi, e^{pN}\Xi'_t\phi_\eta \rangle\!\rangle$.

Applying Gronwall's Lemma, as in [20] or [36], gives the following result.

Corollary 5.5. *Let E, F, G, H be simple processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ and let Ξ be their quantum stochastic integral. Then for $0 \leqslant s \leqslant t \leqslant T$,*

$$|||(\Xi_t - \Xi_s)\phi_\xi|||_p^2 \leqslant 6\alpha(T,\xi)^2 e^t \int_s^t C(\xi; s)\, ds, \quad \xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+), \tag{5.6}$$

where

$$\alpha(T,\xi) = \sup_{0 \leqslant s \leqslant T} \max\{e^{2p}|\xi(s)|^2, e^{2p}|\xi(s)|, |\xi(s)|, 1, e^p\}$$

and

$$C(\xi; s) = |||E_s\phi_\xi|||_p^2 + |||F_s\phi_\xi|||_p^2 + |||G_s\phi_\xi|||_p^2 + |||H_s\phi_\xi|||_p^2.$$

In particular, Ξ is continuous.

By Corollary 5.5 and Proposition 5.2, we can extend the quantum stochastic integral from simple processes to locally square integrable processes, more precisely, let E, F, G, H be locally square integrable processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$, then by Proposition 5.2 there exist four sequences $\{E^{(n)}\}$, $\{F^{(n)}\}$, $\{G^{(n)}\}$, $\{H^{(n)}\}$ of simple processes such that for any $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ and $t \geqslant 0$

$$\begin{aligned}\lim_{n\to\infty} \int_0^t \{&|||(E_s - E_s^{(n)})\phi_\xi|||_p^2 + |||(F_s - F_s^{(n)})\phi_\xi|||_p^2 \\ &+ |||(G_s - G_s^{(n)})\phi_\xi|||_p^2 + |||(H_s - H_s^{(n)})\phi_\xi|||_p^2\}\, ds = 0.\end{aligned}$$

For each $n = 1, 2, \ldots$, put

$$\Xi_t^{(n)} = \int_0^t \left(E^{(n)}\, d\Lambda + F^{(n)}\, dA + G^{(n)}\, dA^* + H^{(n)}\, ds\right).$$

Then for each $t \geqslant 0$ and $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$, by Corollary 5.5, we see that the sequence $\Xi_t^{(n)}\phi_\xi$ converges in $\mathscr{G}_p$. The limit does not depend on the choice of $\left(E^{(n)}, F^{(n)}, G^{(n)}, H^{(n)}\right)$ and so defines a map $\Xi_t \in L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$. The resulting family $\{\Xi_t\}_{t \geqslant 0}$ is an adapted process which is called the quantum stochastic integral of the locally square integrable processes (E, F, G, H) and we write

$$\Xi_t = \int_0^t (E\, d\Lambda + F\, dA + G\, dA^* + H\, ds). \tag{5.7}$$

Then we can show that the equalities in (5.3)–(5.5) and the inequality in (5.6) hold for locally square integrable integrands.

More generally if the adapted processes E, F, G, H in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ satisfy the condition that for any $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ and $t \geqslant 0$

$$\int_0^t |\xi(s)|^2 |||E_s \phi_\xi|||_p^2 \, ds + \int_0^t |\xi(s)| |||F_s \phi_\xi|||_p \, ds + \int_0^t |||G_s \phi_\xi|||_p^2 \, ds + \int_0^t |||H_s \phi_\xi|||_p \, ds < \infty, \tag{5.8}$$

then the integral Ξ in (5.7) is well defined as an adapted process satisfying (5.3).

Remark 5.6. Let E, F, G, H be locally square integrable processes in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$. If

$$\int_0^t (E \, d\Lambda + F \, dA + G \, dA^* + H \, ds) = 0,$$

then it may be shown (cf. [24]) that $E = F = G = H = 0$ except on a Lebesgue measure null set. In this sense, the representation of Ξ_t in (5.7) is unique, i.e., if there exist locally square integrable processes E', F', G', H' in $L(\mathscr{E}_{\mathrm{lb}}, \mathscr{G}_p)$ for which

$$\Xi_t = \int_0^t (E' \, d\Lambda + F' \, dA + G' \, dA^* + H' \, ds), \quad t \geqslant 0,$$

then $E = E'$, $F = F'$, $G = G'$ and $H = H'$ except on a Lebesgue measure null set.

6. Quantum Itô formula

From now on we consider the quantum stochastic processes Ξ in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, thus, $\Xi_t \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ for each $t \geqslant 0$ and $t \mapsto \Xi_t \phi$ is strongly measurable $\mathbb{R}_+ \to \mathscr{G}_q$ for all ϕ in $\mathscr{G}_p$. For $p \geqslant q$ the process is *adapted* if $\Xi_t = \Xi_{t]} \otimes I_{[t}$, where $\Xi_{t]} \in \mathscr{L}(\mathscr{G}_{p;t]}, \mathscr{G}_{q;t]})$ and $I_{[t}$ is the inclusion operator $\mathscr{G}_{p;[t} \to \mathscr{G}_{q;[t}$. For each $p, q \in \mathbb{R}$, let $\mathbf{L}_2(\mathscr{G}_p, \mathscr{G}_q)$ be the class of quadruples $\mathbf{F} = (E, F, G, H)$ of adapted processes in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ satisfying the condition that for any $\phi \in \mathscr{G}_p$ and $t \geqslant 0$

$$\int_0^t |||E_s \dot{\phi}_{s]}|||_q^2 \, ds + \int_0^t |||F_s \dot{\phi}_{s]}|||_q \, ds + \int_0^t |||G_s \phi_{s]}|||_q^2 \, ds + \int_0^t |||H_s \phi_{s]}|||_q \, ds < \infty, \tag{6.1}$$

where $\phi_{t]} = \mathbf{E}_t(\phi)$ and $\dot{\phi}_{t]}$ satisfy the integral representation

$$\phi_{t]} = \langle\!\langle \phi, \phi_0 \rangle\!\rangle + \int_0^t \dot{\phi}_{s]} \, dB_s, \quad t \geqslant 0, \tag{6.2}$$

in particular, for any $\xi \in L^2_{\mathrm{lb}}(\mathbb{R})$ (or H), $\phi_{\xi_{t]}}$ has the following integral representation:

$$\phi_{\xi_{t]}} = \phi_0 + \int_0^t \xi(s) \phi_{\xi_{s]}} \, dB_s, \quad t \geqslant 0.$$

Note that the measurability of $s \mapsto E_s\dot{\phi}_{s]}$ and $s \mapsto F_s\dot{\phi}_{s]}$, required for (6.1) to be well defined, follows from Pettis's Theorem. Then for each $\mathbf{F} \in \mathbf{L}_2(\mathscr{G}_p, \mathscr{G}_p)$, the quantum stochastic integral Ξ_t in (5.7) is well defined on $\mathscr{E}_{\mathrm{lb}}$ and satisfies (5.3). Moreover, by using the same method as is used in [3] we see that for any $\phi \in \mathscr{E}_{\mathrm{lb}}$, Ξ_t satisfies the equation:

$$\Xi_t\phi_{t]} = \int_0^t (\Xi_s + E_s)\dot{\phi}_{s]}\, dB_s + \int_0^t F_s\dot{\phi}_{s]}\, ds + \int_0^t G_s\phi_{s]}\, dB_s + \int_0^t H_s\phi_{s]}\, ds. \tag{6.3}$$

Theorem 6.1. *Let Ξ be an adapted process in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ such that the processes Ξ and Ξ^* admit integral representations of the form (5.7) on $\mathscr{E}_{\mathrm{lb}}$ for some $\mathbf{F} = (E, F, G, H) \in \mathbf{L}_2(\mathscr{G}_p, \mathscr{G}_q)$ and $\mathbf{F}^* = (E^*, G^*, F^*, H^*) \in \mathbf{L}_2(\mathscr{G}_{-q}, \mathscr{G}_{-p})$, respectively. If for all $\phi \in \mathscr{G}_p$ and $\psi \in \mathscr{G}_{-q}$*

$$\int_0^t |||\Xi_s\dot{\phi}_{s]}|||_q^2\, ds < \infty \quad \text{and} \quad \int_0^t |||\Xi_s^*\dot{\psi}_{s]}|||_{-p}^2\, ds < \infty, \tag{6.4}$$

then both integral representations can be extended to $\mathscr{G}_p$ and $\mathscr{G}_{-q}$, respectively. In particular, if $\mathbf{F}$ satisfies the following conditions:

(i) *the map $t \mapsto ||E_t||_{p;q}$ is locally bounded,*
(ii) *the maps $t \mapsto ||F_t||_{p;q}$ and $t \mapsto ||G_t||_{p;q}$ are locally square integrable,*
(iii) *the map $t \mapsto ||H_t||_{p;q}$ is locally integrable,*

where $||\cdot||_{p;q}$ is the operator norm on $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, then the integral representations of Ξ and Ξ^* can be extended to $\mathscr{G}_p$ and *$\mathscr{G}_{-q}$, respectively.*

Proof. For the proof, we use similar arguments to those used in the proof of Theorem 2 in [3]. For any $\phi \in \mathscr{E}_{\mathrm{lb}}$, $\Xi_t\phi_{t]}$ admits the expression in (6.3). Let $R(\phi_{t]})$ be the right-hand side of the expression in (6.3). We observe that for any $\phi, \psi \in \mathscr{E}_{\mathrm{lb}}$

$$\begin{aligned} |\langle\!\langle R(\phi_{t]}), \psi_{t]} \rangle\!\rangle| \leqslant & \int_0^t |\langle\!\langle \Xi_s\dot{\phi}_{s]}, \dot{\psi}_{s]} \rangle\!\rangle|\, ds + \int_0^t |\langle\!\langle E_s\dot{\phi}_{s]}, \dot{\psi}_{s]} \rangle\!\rangle|\, ds \\ & + \int_0^t |\langle\!\langle F_s\dot{\phi}_{s]}, \psi_{s]} \rangle\!\rangle|\, ds \\ & + \int_0^t |\langle\!\langle G_s\phi_{s]}, \dot{\psi}_{s]} \rangle\!\rangle|\, ds \\ & + \int_0^t |\langle\!\langle H_s\phi_{s]}, \psi_{s]} \rangle\!\rangle|\, ds. \end{aligned}$$

Therefore, by Schwartz inequality and the fact that, by (3.2), $\int_0^t |||\dot{\phi}_{s]}|||_p^2\, ds \leqslant e^{-2p}|||\phi_{t]}|||_p^2$, we have

$$
\begin{aligned}
|\langle\!\langle R(\phi_{t]}), \psi_{t]}\rangle\!\rangle| \leqslant & \left[\int_0^t |||G_s^* \dot{\psi}_{s]}|||_{-p}\, ds + \int_0^t |||H_s^* \psi_{s]}|||_{-p}\, ds \right.\\
& + e^{-p}\left(\int_0^t |||\Xi_s^* \dot{\psi}_{s]}|||_{-p}^2\, ds\right)^{1/2}\\
& + e^{-p}\left(\int_0^t |||E_s^* \dot{\psi}_{s]}|||_{-p}^2\, ds\right)^{1/2}\\
& \left. + e^{-p}\left(\int_0^t |||F_s^* \psi_{s]}|||_{-p}^2\, ds\right)^{1/2}\right] |||\phi_{t]}|||_p.
\end{aligned}
$$

Hence the linear map $\phi_{t]} \to \langle\!\langle R(\phi_{t]}), \psi_{t]}\rangle\!\rangle$ is continuous $\mathscr{G}_{p;t]} \to \mathbb{C}$. For any $\phi \in \mathscr{G}_p$, we take a sequence $\{\phi^{(n)}\}_{n=1}^{\infty} \subset \mathscr{E}_{\mathrm{lb}}$ such that $\phi^{(n)}$ converges to ϕ in $\mathscr{G}_p$ and then for any $\psi \in \mathscr{E}_{\mathrm{lb}}$ we have

$$
\begin{aligned}
& |\langle\!\langle \Xi_t \phi_{t]} - R(\phi_{t]}), \psi_{t]}\rangle\!\rangle|\\
& \quad \leqslant |\langle\!\langle \Xi_t \phi_{t]} - \Xi_t \phi_{t]}^{(n)}, \psi_{t]}\rangle\!\rangle| + |\langle\!\langle \Xi_t \phi_{t]}^{(n)} - R(\phi_{t]}^{(n)}), \psi_{t]}\rangle\!\rangle|\\
& \qquad + |\langle\!\langle R(\phi_{t]} - \phi_{t]}^{(n)}), \psi_{t]}\rangle\!\rangle|\\
& \quad \leqslant M|||\psi_{t]}|||_{-q}|||\phi_{t]} - \phi_{t]}^{(n)}|||_p + |\langle\!\langle R(\phi_{t]} - \phi_{t]}^{(n)}), \psi_{t]}\rangle\!\rangle|
\end{aligned}
$$

for some $M \geqslant 0$. Therefore, $R(\phi_{t]})$ can be extended to $\mathscr{G}_p$ and $\Xi_t \phi_{t]} = R(\phi_{t]})$. Note that under condition (6.4), Eq. (6.3) is equivalent to (5.7) on $\mathscr{E}_{\mathrm{lb}}$. This proves the first statement. The proof of the second statement is obvious. □

The following theorem is a generalization of the Itô formula for composition of processes proved by Attal and Meyer [3] to a wider class of certain unbounded processes.

Theorem 6.2. *Let Ξ and Ξ' be adapted processes in $\mathscr{L}(\mathscr{G}_q, \mathscr{G}_r)$ and $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, respectively, satisfying the hypothesis of Theorem* 6.1 *and admitting the representations*:

$$
\begin{aligned}
\Xi_t &= \int_0^t (E\, d\Lambda + F\, dA + G\, dA^* + H\, ds),\\
\Xi'_t &= \int_0^t (E'\, d\Lambda + F'\, dA + G'\, dA^* + H'\, ds)
\end{aligned}
$$

on $\mathscr{E}_{\rm lb}$ *with some* $\mathbf{F}\in\mathbf{L}_2(\mathscr{G}_p,\mathscr{G}_q)$ *and* $\mathbf{F}'\in\mathbf{L}_2(\mathscr{G}_p,\mathscr{G}_q)$. *Then we have*

$$\begin{aligned}\Xi_t\Xi_t' = &\int_0^t (E\Xi'\,d\Lambda + F\Xi'\,dA + G\Xi'\,dA^* + H\Xi'\,ds)\\ &+ \int_0^t (\Xi E'\,d\Lambda + \Xi F'\,dA + \Xi G'\,dA^* + \Xi H'\,ds)\\ &+ \int_0^t (EE'\,d\Lambda + FE'\,dA + EG'\,dA^* + FG'\,ds).\end{aligned} \tag{6.5}$$

Proof. By similar arguments to those used in the proof of Theorem 4 in [3], the proof is now straightforward. □

7. Integral representation of quantum martingales

In the following, we consider quantum martingales Ξ in $\mathscr{L}(\mathscr{G}_p,\mathscr{G}_q)$. Thus, for any $0\leqslant s\leqslant t$, $\phi_{s]}\in\mathscr{G}_{p;s]}$ and $\psi_{s]}\in\mathscr{G}_{-q;s]}$

$$\langle\!\langle \Xi_t\phi_{s]},\psi_{s]}\rangle\!\rangle = \langle\!\langle \Xi_s\phi_{s]},\psi_{s]}\rangle\!\rangle. \tag{7.1}$$

The following definition of regular martingale is a simple modification of the definition of bounded regular martingale in [37].

Definition 7.1. A quantum martingale Ξ in $\mathscr{L}(\mathscr{G}_p,\mathscr{G}_q)$ is said to be *regular with respect to a Radon measure* m on $\mathbb{R}_+$, or simply *regular* if for all $0\leqslant v<u$ and $\phi\in\mathscr{G}_{p;v]}$, $\psi\in\mathscr{G}_{-q;v]}$

$$|||(\Xi_u-\Xi_v)\phi|||_q^2\leqslant|||\phi|||_p^2 m([v,u]),\quad |||(\Xi_u^*-\Xi_v^*)\psi|||_{-p}^2\leqslant|||\psi|||_{-q}^2 m([v,u]). \tag{7.2}$$

Let l,m be non-negative integers. Then by similar arguments to those used in the proof of Lemma 4.1, we can prove that for any $p\in\mathbb{R}$ and $q>0$ there exists a constant $C\geqslant 0$ such that for all $0\leqslant v<u$ and $\phi\in\mathscr{G}_{p+q;v]}$

$$|||((A_u^*)^l A_u^m-(A_v^*)^l A_v^m)\phi|||_p^2\leqslant C|||\phi|||_{p+q}^2(u^l-v^l)v^m \tag{7.3}$$

and

$$|||(\Lambda_u^{\diamond l}-\Lambda_v^{\diamond l})\phi|||_p^2=0. \tag{7.4}$$

Since $(u^l-v^l)v^m\leqslant u^{l+m}-v^{l+m}$ for any $0\leqslant v<u$, the processes $\{(A_t^*)^l A_t^m\}_{t\geqslant 0}$ and $\{\Lambda_t^{\diamond l}\}_{t\geqslant 0}$ are regular martingales in $\mathscr{L}(\mathscr{G}_{p+q},\mathscr{G}_p)$. Also, we can see that the processes $\{\mathrm{wexp}\, Q_{a,b}(t)\}_{t\geqslant 0}$ and $\{\mathrm{wexp}\,\Lambda_t\}_{t\geqslant 0}$ are regular martingales in $\mathscr{L}(\mathscr{G}_{p+q},\mathscr{G}_p)$ where in the latter case $q>\lambda_0$, the positive solution of $e^{-\lambda}=\lambda^2$.

Now, we state the main theorem in this section which generalizes the stochastic integral representation of regular bounded quantum martingales due to Parthasarathy and Sinha [37] to certain regular (unbounded) quantum martingales.

Theorem 7.2. *Let Ξ be a martingale in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, regular with respect to a Radon measure m on $\mathbb{R}_+$. Then there exist adapted processes E, F, G in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ and $\lambda \in \mathbb{C}$ such that*

$$\Xi_t = \lambda I + \int_0^t (E\,d\Lambda + F\,dA + G\,dA^*) \tag{7.5}$$

on $\mathscr{E}_{\mathrm{lb}}$, and $||E_{(\cdot)}||_{p;q}$ is locally bounded and

$$\max\{||F_s||_{p;q}^2, ||G_s||_{p;q}^2\} \leqslant m'_{\mathrm{ac}}(s) \quad \textit{for all } s,$$

where m'_{ac} denotes the density of the absolutely continuous part of m. Such a triple (E, F, G) is unique in the sense of Remark 5.6. *Conversely, if a process Ξ in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ admits the integral representation* (7.5) *with the adapted processes E, F, G in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ such that $||F_{(\cdot)}||_{p;q}$ and $||G_{(\cdot)}||_{p;q}$ are locally square integrable, then Ξ is regular.*

By Theorem 6.1, the integral representation (7.5) can be extended to all of $\mathscr{G}_p$. For the proof of Theorem 7.2, we will use similar arguments to those used in [29,37].

Proposition 7.3. *Let Ξ be a quantum martingale in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. If Ξ has the integral representation* (7.5) *with $\lambda \in \mathbb{C}$ and quadruple $(E, F, G, 0) \in \mathbf{L}_2(\mathscr{G}_p, \mathscr{G}_q)$ satisfying $(E^*, G^*, F^*, 0) \in \mathbf{L}_2(\mathscr{G}_{-q}, \mathscr{G}_{-p})$ with $||F_{(\cdot)}||_{p;q}$ and $||G_{(\cdot)}||_{p;q}$ locally square integrable, then Ξ is regular with respect to an absolutely continuous Radon measure m.*

Proof. It follows from the definition of quantum martingale and (7.5) that Ξ^* is a quantum martingale in $\mathscr{L}(\mathscr{G}_{-q}, \mathscr{G}_{-p})$ with integral representation

$$\Xi_t^* = \lambda I + \int_0^t (E^*\,d\Lambda + G^*\,dA + F^*\,dA^*).$$

Note that for any $0 \leqslant s < t$ and $\phi_{s]} \in \mathscr{G}_{p;s]}$, $\psi_{s]} \in \mathscr{G}_{q;s]}$

$$\langle\!\langle \Xi_t \phi_{s]}, \psi_{s]} \rangle\!\rangle_q = \langle\!\langle \Xi_s \phi_{s]}, \psi_{s]} \rangle\!\rangle_q.$$

It follows that

$$|||(\Xi_t - \Xi_a)\phi_{a]}|||_q^2 = |||\Xi_t \phi_{a]}|||_q^2 - |||\Xi_a \phi_{a]}|||_q^2. \tag{7.6}$$

Therefore, by applying (5.5), we obtain that for any $0 \leqslant a < t$ and $\phi_{a]} \in \mathscr{E}_{\mathrm{lb};a]}$

$$|||(\Xi_t - \Xi_a)\phi_{a]}|||_q^2 = e^{2q} \int_a^t |||G_s \phi_{a]}|||_q^2 \, ds \leqslant e^{2q} |||\phi_{a]}|||_p^2 \int_a^t ||G_s||_{p;q}^2 \, ds. \tag{7.7}$$

Similarly, for any $0 \leqslant a < t$ and $\psi_{a]} \in \mathscr{E}_{\mathrm{lb};a]}$, we have

$$|||(\Xi_t^* - \Xi_a^*)\psi_{a]}|||_{-p}^2 \leqslant e^{-2p} |||\psi_{a]}|||_{-q}^2 \int_a^t ||F_s^*||_{-q;-p}^2 \, ds. \tag{7.8}$$

Now, we define a Radon measure m on $\mathbb{R}_+$ by

$$m([a,b]) = \int_a^b (e^{2q} ||G_s||_{p;q}^2 + e^{-2p} ||F_s^*||_{-q;-p}^2) \, ds \quad \text{for all } 0 \leqslant a \leqslant b < \infty.$$

Therefore, by (7.7), (7.8) and the density of $\mathscr{E}_{\mathrm{lb};a]}$ in $\mathscr{G}_{p;a]}$ and $\mathscr{G}_{-q;a]}$, we see that Ξ is regular with respect to the absolutely continuous Radon measure m. $\square$

Remark 7.4. Let Ξ be the martingale in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, regular with respect to the Radon measure m, then:

(i) For any $t > a$,

$$||\Xi_t||_{p;q} \geqslant \sup_{|||\phi_{a]}|||_p = 1} |||\Xi_t \phi_{a]}|||_q \geqslant \sup_{|||\phi_{a]}|||_p = 1} |||\Xi_a \phi_{a]}|||_q \geqslant ||\Xi_a||_{p;q},$$

where we used (7.6) for the second inequality. Therefore, $||\Xi_{(\cdot)}||_{p;q}$ is non-decreasing.

(ii) Since $\{\Xi_t \phi\}_{t \geqslant a} \subset \mathscr{G}_q$ and $\{\Xi_t^* \psi\}_{t \geqslant a} \subset \mathscr{G}_{-p}$ are (generalized) martingales for each $\phi \in \mathscr{G}_{p;a]}$ and $\psi \in \mathscr{G}_{-q;a]}$, by Theorem 3.2 there exist $\{\Phi(t,\phi)\}_{t \geqslant a} \subset \mathscr{G}_q$ and $\{\Psi(t,\psi)\}_{t \geqslant a} \subset \mathscr{G}_{-p}$ such that

$$(\Xi_t - \Xi_a)\phi = \int_a^t \Phi(s,\phi) \, dB_s \quad \text{and} \quad (\Xi_t^* - \Xi_a^*)\psi = \int_a^t \Psi(s,\psi) \, dB_s.$$

On the other hand, by (3.2) we have that for all $0 \leqslant a \leqslant b < t$

$$\int_b^t e^{2q} |||\Phi(t,\phi)|||_q^2 \, ds = |||(\Xi_t - \Xi_b)\phi|||_q^2 \leqslant |||\phi|||_p^2 m([b,t]) \tag{7.9}$$

and

$$\int_b^t e^{-2p} |||\Psi(t,\psi)|||_{-p}^2 \, ds = |||(\Xi_t^* - \Xi_b^*)\psi|||_{-p}^2 \leqslant |||\psi|||_{-q}^2 m([b,t]).$$

These imply that m in (7.2) can be replaced by its absolutely continuous part m_{ac} and without lose of generality, we can assume that m is an absolutely continuous Radon measure.

Proposition 7.5. *Let Ξ be a martingale in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, regular with respect to an absolutely continuous Radon measure m on $\mathbb{R}_+$ with density $m'(t)$. Then there are adapted processes G in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ and F^* in $\mathscr{L}(\mathscr{G}_{-q}, \mathscr{G}_{-p})$ such that*

(i) *for any $a<t$, $\phi \in \mathscr{G}_{p;a]}$ and $\psi \in \mathscr{G}_{-q;a]}$*

$$(\Xi_t - \Xi_a)\phi = \int_a^t G_s\phi\, dB_s \quad \text{and} \quad (\Xi_t^* - \Xi_a^*)\psi = \int_a^t F_s^*\psi\, dB_s;$$

(ii) *for any s, $||G_s||_{p;q}^2 \leqslant e^{-2q} m'(s)$ and $||F_s^*||_{-q;-p}^2 \leqslant e^{2p} m'(s)$.*

Proof. For fixed $a>0$ and $\phi \in \mathscr{G}_{p;a]}$, there exists an adapted process $\{\Phi_a(t,\phi)\}_{t\geqslant a} \subset \mathscr{G}_q$ such that

$$(\Xi_t - \Xi_a)\phi = \int_a^t \Phi_a(s,\phi)\, dB_s, \quad t>a. \tag{7.10}$$

Then by the uniqueness of integral representations, we may define (a.e.) an adapted process by

$$\Phi(s,\phi) = \Phi_a(s,\phi), \quad s>a, \ \ \phi \in \mathscr{G}_{p;a]}$$

and then by (3.2), (7.2) and (7.10), we have

$$|||\Phi(s,\phi)|||_q^2 \leqslant e^{-2q} |||\phi|||_p^2 m'(s), \quad \text{a.e. } s>a. \tag{7.11}$$

Consider the index set $I := \{\gamma = \left(r_1, \ldots, r_k; \xi^{(1)}, \ldots, \xi^{(k)}\right); r_j \in \mathbb{Q} + i\mathbb{Q}, \xi^{(j)} \in D, k = 1, 2, \ldots\}$, where D is a countable dense subset of H. For each $\gamma = \left(r_1, \ldots, r_k; \xi^{(1)}, \ldots, \xi^{(k)}\right) \in I$, we define

$$\vartheta(a,s,\gamma) = \begin{cases} 0 & \text{if } s\leqslant a, \\ \left|\left|\left|\sum_{j=1}^k r_j \Phi\left(s, \phi_{\xi_{a]}^{(j)}}\right)\right|\right|\right|_q^2 - e^{-2q} m'(s) \left|\left|\left|\sum_{j=1}^k r_j \phi_{\xi_{a]}^{(j)}}\right|\right|\right|_p^2 & \text{if } s>a. \end{cases}$$

Since $\phi \mapsto \Phi(s,\phi)$ is linear, it follows from (7.11) that there exists a null set $F \subset \mathbb{R}_+$ such that

$$\left|\left|\left|\sum_{j=1}^k r_j \Phi\left(s, \phi_{\xi_{a]}^{(j)}}\right)\right|\right|\right|_q^2 \leqslant e^{-2q} m'(s) \left|\left|\left|\sum_{j=1}^k r_j \phi_{\xi_{a]}^{(j)}}\right|\right|\right|_p^2 \tag{7.12}$$

for any $a \in \mathbb{Q}$ with $a>0$, $\gamma \in I$ and $s \notin F$, $s>a$. Now, for each $s>0$ we define an operator $G_{s]}$ by

$$G_{s]}\phi_{\xi_{a]}} = \begin{cases} \Phi(s, \phi_{\xi_{a]}}) & \text{if } s \notin F, \\ 0 & \text{otherwise} \end{cases} \tag{7.13}$$

for $a \in \mathbb{Q}$ with $a > 0$, $s > a$, $\xi \in D$ and extended by linearity to $\mathscr{E}_{\mathrm{lb}}$. Since

$$\left\{ \sum_{j=1}^{k} r_j \phi_{\xi_{a]}^{(j)}} ; \gamma \in I \right\} \quad \text{and} \quad \bigcup_{0 < a < s, a \in \mathbb{Q}} \mathscr{G}_{p;a]}$$

are dense subsets of $\mathscr{G}_{p;a]}$ and $\mathscr{G}_{p;s]}$, respectively, by (7.12) the operator $G_{s]}$ can be extended to a bounded operator $\mathscr{G}_{p;s]} \to \mathscr{G}_{q;s]}$ which is still denoted by $G_{s]}$. Then, for each $\phi \in \mathscr{G}_{p;a]}$,

$$G_{s]}\phi = \Phi(s, \phi)\text{a.e. } s > a \quad \text{and} \quad \|G_{s]}\|_{p;q}^2 \leqslant e^{-2q} m'(s) \text{ for any } s.$$

Since $G_{s]}$ can be extended uniquely by ampliation of $\mathscr{G}_p$ and then be denoted by G_s, we have an adapted process G in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. Similarly, we can define an adapted process F^* in $\mathscr{L}(\mathscr{G}_{-q}, \mathscr{G}_{-p})$ such that G and F^* satisfy (i) and (ii). □

It is clear from the proof of Proposition 7.5 that the adapted processes G and F^* are uniquely determined (modulo a Lebesgue null set).

Let Ξ be a regular martingale in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ satisfying the hypothesis in Proposition 7.5 and let G and F^* be the processes given in Proposition 7.5. Then for any $t \geqslant 0$ we define an operator $N_t : \mathscr{G}_{p;t]} \to \mathscr{G}_{q;t]}$ by

$$N_t \phi_{t]} = \Xi_t \phi_{t]} - \int_0^t \Xi_u \dot{\phi}_{u]} \, dB_u - \int_0^t F_u \dot{\phi}_{u]} \, du - \int_0^t G_u \phi_{u]} \, dB_u, \quad \phi \in \mathscr{G}_p. \tag{7.14}$$

By (3.2) and direct computation using properties in (ii) of Proposition 7.5 and the fact that $\|\Xi_{(\cdot)}\|_{p;q}$ is non-decreasing, we can see that for all $t \geqslant 0$ this indeed defines an operator $N_t \in \mathscr{L}(\mathscr{G}_{p;t]}, \mathscr{G}_{q;t]})$.

Proposition 7.6. *Let N_t be defined as in* (7.14). *Then we have*

(i) *for any $s < t$ and $\phi \in \mathscr{G}_{p;s]}$, $(N_t - N_s)\phi = 0$;*
(ii) *$N_{(\cdot)}$ transforms martingales into martingales, i.e. for any $s < t$, $\mathbf{E}_s(N_t \phi_{t]}) = N_s \phi_{s]}$;*
(iii) *for any $s < t$, $\mathbf{E}_s N_t = N_t \mathbf{E}_s$.*

Proof. (i) It is obvious that for any $\phi \in \mathscr{G}_{p;s]}$, $\chi_{[s,t]}(u)\dot{\phi}_{u]} = 0$. On the other hand, by (i) in Proposition 7.5, we have

$$(\Xi_t - \Xi_s)\phi = \int_s^t G_u \phi \, dB_u = \int_s^t G_u \phi_{u]} \, dB_u.$$

Therefore, the result follows from the definition of N_t.

(ii) By (6.2) and (i) in Proposition 7.5, we see that for any $s<t$

$$\langle\langle N_t\phi_{t]}, \phi_{\xi_{s]}}\rangle\rangle = \langle\langle N_s\phi_{s]}, \phi_{\xi_{s]}}\rangle\rangle, \quad \phi\in\mathscr{G}_p, \ \xi\in H.$$

The result follows.

(iii) The proof is obvious from (i) and (ii). □

Remark 7.7. For any given $T\geqslant 0$, by (6.2) and the Itô isometry, we define an isometry between $\mathscr{G}_{p,T]}$ and $\mathbb{C}\oplus L^2_{\mathrm{a}}([0,T],\mathscr{G}_{p;T]})$ (the space of all $\mathscr{G}_p$-valued adapted processes on $[0,T]$) by

$$\iota_T : \phi_{T]}\in\mathscr{G}_{p;T]}\mapsto \langle\langle\phi_{T]},\phi_0\rangle\rangle + e^p\dot{\phi}_{\cdot]}\in\mathbb{C}\oplus L^2_{\mathrm{a}}([0,T],\mathscr{G}_{p;T]}).$$

Let $\Xi\in\mathscr{L}(L^2([0,T],\mathscr{G}_p),L^2([0,T],\mathscr{G}_q))$ commute with multiplication operators by (scalar-valued) bounded Borel functions. Then by using a lifting theorem (see [29, pp. 293, 295], where we assume the continuum hypothesis), there exists a strongly measurable family of operators $\{K_t\}\subset\mathscr{L}(\mathscr{G}_p,\mathscr{G}_q)$ with $||K_t||_{p;q}\leqslant k$ for a.e. t and some constant $k\geqslant 0$ such that $(\Xi\phi)_t = (K_t\phi_t)$. Also, K_t does not depend on the choice of the interval $[0,T]$ containing t.

Proposition 7.8. *Let N_t be defined as in* (7.14). *Then there exists an adapted process E in $\mathscr{L}(\mathscr{G}_p,\mathscr{G}_q)$ such that for any $\phi\in\mathscr{G}_p$*

$$N_t\phi_{t]} = \lambda\mathbf{E}(\phi)\phi_0 + \int_0^t E_s\dot{\phi}_{s]}\,ds, \quad t\geqslant 0,$$

where $\lambda\in\mathbb{C}$ with $\Xi_0\phi_0=\lambda\phi_0$. In this case, the map $t\mapsto ||E_t||_{p;q}$ is locally bounded.

Proof. By (ii) in Proposition 7.6 and Theorem 3.2, for any $\phi\in\mathscr{G}_p$ there exists a unique adapted process ψ in $\mathscr{G}_q$ and $\lambda\in\mathbb{C}$ such that

$$N_t\phi_{t]} = N_t\left(\mathbf{E}(\phi)\phi_0 + \int_0^t \dot{\phi}_{s]}\,dB_s\right) = \lambda\mathbf{E}(\phi)\phi_0 + \int_0^t \psi_s\,dB_s, \quad t\geqslant 0. \tag{7.15}$$

Now, we consider the mapping $\tilde{N}$ from $L^2_{\mathrm{a}}([0,T],\mathscr{G}_p)$ into $L^2_{\mathrm{a}}([0,T],\mathscr{G}_q)$ defined by $\tilde{N}:(\dot{\phi}_{s]})\mapsto(\psi_s)$, where $\dot{\phi}_{s]}$ and ψ_s are given as in (7.15). Then $\tilde{N}$ is bounded. In fact, by (3.2), there exists a constant $C\geqslant 0$ such that

$$\begin{aligned}\int_0^t |||\psi_s|||_q^2\,ds &= e^{-2q}\left|\left|\left|N_t\left(\int_0^t \dot{\phi}_{s]}\,dB_s\right)\right|\right|\right|_q^2\\ &\leqslant Ce^{-2q}\left|\left|\left|\int_0^t \dot{\phi}_{s]}\,dB_s\right|\right|\right|_p^2\\ &= Ce^{2(p-q)}\int_0^t |||\dot{\phi}_{s]}|||_p^2 ds.\end{aligned}$$

On the other hand, we can see that $\tilde{N}$ commutes with multiplication operators by bounded Borel functions on $[0, T]$. Let P be the previsible (adapted) projection operator. Then $\tilde{N}$ can be extended to an operator in $\mathscr{L}(L^2([0, T], \mathscr{G}_p), L^2([0, T], \mathscr{G}_q))$ by composing with P. Then $\tilde{N} \circ P$ also commutes with multiplication by bounded Borel functions. Therefore, by Remark 7.7, there exists a strongly measurable family of operators $\{K_t\} \subset \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ with $\|K_t\|_{p;q} \leqslant k$ for a.e. t and some constant $k \geqslant 0$ such that $(\tilde{N} \circ P\phi)_t = (K_t \phi_t)$. Put $E_t = \mathbf{E}_t(K_t)$ for each $t \geqslant 0$ to ensure adaptedness. Finally, we have

$$N_t \phi_{t]} = \lambda \mathbf{E}(\phi)\phi_0 + \int_0^t E_s \dot{\phi}_{s]}\, ds, \quad t \geqslant 0.$$

This completes the proof. $\square$

Proof of Theorem 7.2. From (7.14) and Proposition 7.8, we have

$$\begin{aligned} &\Xi_t \phi_{t]} \\ &\quad = \lambda \mathbf{E}(\phi)\phi_0 + \int_0^t \Xi_u \dot{\phi}_{u]}\, dB_u + \int_0^t E_s \dot{\phi}_{s]}\, ds + \int_0^t F_u \dot{\phi}_{u]}\, du + \int_0^t G_u \phi_{u]}\, dB_u, \quad \phi \in \mathscr{G}_p. \end{aligned}$$

Since this equation is equivalent to Eq. (7.5) on $\mathscr{E}_{\mathrm{lb}}$, the first part is proved. The uniqueness is obvious from the definitions of E, F, G. The converse is proved in Proposition 7.3. $\square$

Example 7.9. Let l, m be non-negative integers. Then by repeated application of the Itô formula (6.5),

$$(A_t^*)^l A_t^m = m \int_0^t (A_s^*)^l A_s^{m-1}\, dA_s + l \int_0^t (A_s^*)^{l-1} A_s^m dA_s^*.$$

Therefore, by (4.4)

$$\begin{aligned} Q_{a,b}^{\diamond n}(t) &= \sum_{l+m=n} \frac{n!}{l!m!} a^m b^l \left(m \int_0^t (A_s^*)^l A_s^{m-1} dA_s + l \int_0^t (A_s^*)^{l-1} A_s^m\, dA_s^* \right) \\ &= n \int_0^t Q_{a,b}^{\diamond (n-1)}(s)\, dQ_{a,b}(s). \end{aligned}$$

Hence

$$\operatorname{wexp} Q_{a,b}(t) = I + \int_0^t \operatorname{wexp} Q_{a,b}(s)\, dQ_{a,b}(s).$$

Similarly,

$$\Lambda_t^{\diamond l} = l\int_0^t \Lambda_s^{\diamond (l-1)}\, d\Lambda_s, \quad \Lambda_t^l = \int_0^t ((1+\Lambda_s)^l - \Lambda_s^l)\, d\Lambda_s$$

and

$$\operatorname{wexp} \Lambda_t = I + \int_0^t \operatorname{wexp} \Lambda_s\, d\Lambda_s, \quad e^{\Lambda_t} = I + (e-1)\int_0^t e^{\Lambda_s}\, d\Lambda_s.$$

8. Integral representation of quantum semimartingales

An adapted process Ξ in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ is called a *regular* (*quantum*) *semimartingale* if there is an absolutely continuous Radon measure m on $\mathbb{R}_+$ such that for any $r<s<t$ and $\phi \in \mathscr{G}_{p;r]}$, $\psi \in \mathscr{G}_{-q;r]}$

$$|||(\Xi_t - \Xi_s)\phi|||_q^2 \leqslant |||\phi|||_p^2 m([s,t]), \tag{8.1}$$

$$|||(\Xi_t^* - \Xi_s^*)\psi|||_{-p}^2 \leqslant |||\psi|||_{-q}^2 m([s,t]), \tag{8.2}$$

$$|||(\mathbf{E}_s\Xi_t - \Xi_s)\phi|||_q \leqslant |||\phi|||_p m([s,t]). \tag{8.3}$$

This definition of regular semimartingale is a simple modification of the definition of bounded regular semimartingale in [1].

Now, we define the space $\mathscr{S}_{p,q}$ of adapted processes Ξ in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ admitting the integral representation

$$\Xi_t = \lambda I + \int_0^t (E\, d\Lambda + F\, dA + G\, dA^* + H\, ds) \tag{8.4}$$

on $\mathscr{E}_{\mathrm{lb}}$ with $\lambda \in \mathbb{C}$ and adapted processes (E, F, G, H) in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ such that $s \mapsto ||F_s||_{p;q}$ and $s \mapsto ||G_s||_{p;q}$ are locally square integrable, $s \mapsto ||E_s||_{p;q}$ is locally bounded and $s \mapsto ||H_s||_{p;q}$ is locally integrable.

Theorem 8.1. *An adapted process Ξ in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ belongs to $\mathscr{S}_{p,q}$ if and only if Ξ is a regular semimartingale.*

Proof. If Ξ is an element of $\mathscr{S}_{p,q}$ with integral representation (8.4), then for any $r<s<t$ and $\phi \in \mathscr{G}_{p;r]}$, we have

$$|||(\Xi_t - \Xi_s)\phi|||_q^2 \leqslant 2\left(|||(M_t - M_s)\phi|||_q^2 + \left[\int_s^t ||H_u||_{p;q}\, du\right]^2 |||\phi|||_p^2\right), \tag{8.5}$$

where M is the quantum martingale with the integral representation

$$M_t = \lambda I + \int_0^t (E\,d\Lambda + F\,dA + G\,dA^*).$$

Hence by Proposition 7.3 and (8.5), it easily follows that (8.1)–(8.3) hold, i.e., Ξ is regular.

Conversely, let Ξ be a regular semimartingale. Then by (8.3) and Theorem 3.3, for any $\phi_{r]} \in \mathscr{G}_{p;r]}$ the (generalized) process $\{\Xi_t \phi_{r]}\}_{t \geqslant r} \subset \mathscr{G}_q$ is a regular semimartingale. Hence there exist a martingale $\{L_t\}_{t\geqslant r} \subset \mathscr{G}_q$ and an adapted process $\{\Psi_t(\phi_{r]})\}_{t\geqslant r}$ in $\mathscr{G}_q$ satisfying $|||\Psi_t(\phi_{r]})|||_q \leqslant |||\phi_{r]}|||_p m'(t)$ a.e. t such that for any $r<s<t$

$$\Xi_t \phi_{r]} - \Xi_s \phi_{r]} = L_t - L_s + \int_s^t \Psi_u(\phi_{r]})\,du.$$

Now, for each $r<u$, we define a continuous linear operator H_u^r from $\mathscr{G}_{p;r]}$ into $\mathscr{G}_q$ by

$$H_u^r \phi_{r]} = \begin{cases} \Psi_u(\phi_{r]}) & \text{if } m'(u) \text{ exists}, \\ 0 & \text{if } m'(u) \text{ does not exist}. \end{cases}$$

Then we get a mapping $u \mapsto H_u^r \phi_{r]}$ from (r, ∞) into $\mathscr{G}_q$ satisfying $|||H_u^r \phi_{r]}|||_q \leqslant |||\phi_{r]}|||_p m'(u)$. Note that for $r<s<u$, H_u^r is the restriction of H_u^s to $\mathscr{G}_{p;r]}$. Hence we define an operator $H_{u]}$ on $\bigcup_{r<u} \mathscr{G}_{p;r]}$ by

$$H_{u]}\phi = H_u^r \phi, \quad \phi \in \mathscr{G}_{p;r]}$$

and extend it to $\mathscr{G}_{p;u]}$ by continuity. Then it is immediate that $H_{u]} \in \mathscr{L}(\mathscr{G}_{p;u]}, \mathscr{G}_{q;u]})$. Finally, in the factorization $\mathscr{G}_p = \mathscr{G}_{p;u]} \otimes \mathscr{G}_{p;[u}$, put H_u as $H_{u]} \otimes I_{[u}$ so that H is an adapted process in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. Moreover, $\int_0^t H_u\,du \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ for any $t \geqslant 0$. Put

$$\Upsilon_t = \Xi_t - \int_0^t H_u\,du, \quad t \geqslant 0.$$

Obviously, Υ is a martingale in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. It is easily shown that Υ satisfies the condition in (7.2) and so is a regular martingale. Thus by applying Theorem 7.2, we complete the proof. $\square$

Example 8.2. By using the commutation relation $[A_t, A_t^*] = t$, we obtain that for any $n \geqslant 1$

$$Q_{a,b}^n(t) = \sum_{2j+k+l=n} \frac{n!}{2^j j! k! l!} a^{j+l} b^{j+k} t^j A_t^{*k} A_t^l, \quad t \in \mathbb{R}^+. \tag{8.6}$$

Therefore, by Theorem 4.2, for any $\phi \in \mathcal{G}$ and $p \in \mathbb{R}$, $q > 0$ we have

$$\begin{aligned} |||Q^n_{a,b}(t)\phi|||_p &\leqslant \sum_{2j+k+l=n} \frac{n!}{2^j j!k!l!} |a|^{j+l}|b|^{j+k} t^j |||A_t^{*k} A_t^l \phi|||_p \\ &\leqslant \sum_{2j+k+l=n} \frac{n!}{2^j j!k!l!} |a|^{j+l}|b|^{j+k} t^j t^{(k+l)/2} e^{pk-(p+q)l+q/2} l^{l/2} k^{k/2} D_q^{(l+k)/2} |||\phi|||_{p+q} \\ &\leqslant e^{q/2} |||\phi|||_{p+q} \left(\sum_{2j+k+l=n} \frac{n!}{j!k!l!} \alpha^j \beta^l \gamma^k l^{l/2} k^{k/2} \right), \end{aligned}$$

where the non-negative constants α, β and γ are given by

$$\alpha = \frac{|a||b|t}{2}, \quad \beta = |a|(tD_q)^{1/2} e^{-(p+q)}, \quad \gamma = |b|(tD_q)^{1/2} e^p.$$

Hence for any $p \in \mathbb{R}$ and $q > 0$ we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} ||Q^n_{a,b}(t)||_{p+q;p} \leqslant e^{q/2} \left(\sum_{n=0}^{\infty} \sum_{2j+k+l=n} \frac{1}{j!k!l!} \alpha^j \beta^l \gamma^k l^{l/2} k^{k/2} \right). \tag{8.7}$$

Since by Stirling's formula we can see that the series on the right-hand side of (8.7) converges, the operator $\exp Q_{a,b}(t)$ is well defined and $\exp Q_{a,b}(t)$ belongs to $\mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$. Also, by similar arguments to those used in the proof of Lemma 4.1, we see that the adapted processes $\{Q^n_{a,b}(t)\}_{t \geqslant 0}$ and $\{e^{Q_{a,b}(t)}\}_{t \geqslant 0}$ are regular semimartingales. By induction, we see that for any $n \geqslant 1$

$$Q^n_{a,b}(t) = n \int_0^t Q^{n-1}_{a,b}(s)\, dQ_{a,b}(s) + \frac{1}{2} n(n-1)ab \int_0^t Q^{n-2}_{a,b}(s)\, ds.$$

Therefore, we have the following integral representation:

$$e^{Q_{a,b}(t)} = I + \int_0^t e^{Q_{a,b}(s)}\, dQ_{a,b}(s) + \frac{1}{2} ab \int_0^t e^{Q_{a,b}(s)}\, ds.$$

On the other hand, by (8.6),

$$Q^n_{a,b}(t) = n! \sum_{2j+m=n} \frac{1}{j!m!} \left(\frac{abt}{2} \right)^j (aA_t + bA_t^*)^{\diamond m}.$$

Therefore,

$$e^{Q_{a,b}(t)} = \sum_{n=0}^{\infty} \sum_{j=0}^{[n/2]} \frac{1}{j!(n-2j)!} \left(\frac{abt}{2}\right)^j (aA_t + bA_t^*)^{\diamond (n-2j)}$$
$$= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{j!n!} \left(\frac{abt}{2}\right)^j (aA_t + bA_t^*)^{\diamond n}.$$

Hence

$$\operatorname{wexp} Q_{a,b}(t) = e^{Q_{a,b}(t) - \frac{abt}{2}}.$$

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